

# On the paper: Numerical radius preserving linear maps on Banach algebras

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**Abstract.** In this note, we give a counterexample disproving two results in the above paper.

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Let  $(A, \|\cdot\|)$  be a complex normed algebra with an identity  $e$ . Let  $D(A, e) = \{f \in A' : f(e) = \|f\| = 1\}$ , where  $A'$  is the dual space of  $A$ . The elements of  $D(A, e)$  are called normalized states on  $A$ . For  $a \in A$ , let  $V(A, a) = \{f(a) : f \in D(A, e)\}$ ,  $V(A, a)$  is called the numerical range of  $a$ . Let  $sp(a)$  be the spectrum of  $a \in A$ , and let  $co(sp(a))$  be the convex hull of  $sp(a)$ . We denote by  $M(A)$  the set of all non-zero continuous multiplicative linear functionals on  $A$ .

In [4], F. Golfarshchi and A. A. Khalilzadeh proved the following results:

[4, **Theorem 2**]. Let  $A$  be a unital complex Banach algebra, and let  $f$  be a linear functional on  $A$ . Then  $f$  is a normalized state on  $A$  if and only if  $f(a) \in co(sp(a))$  for all  $a \in A$ .

[4, **Theorem 3**]. Let  $A$  be a unital commutative complex Banach algebra. Then each extreme normalized state on  $A$  is multiplicative.

**Counterexample.** Let  $(A, \|\cdot\|)$  be a non-zero commutative radical complex Banach algebra [6, p.316]. Let  $A_e = \{a + \lambda e : a \in A, \lambda \in C\}$  be the unitization of  $A$  with the identity  $e$ , and the norm  $\|a + \lambda e\|_1 = \|a\| + |\lambda|$  for all  $a + \lambda e \in A_e$ .  $(A_e, \|\cdot\|_1)$  is a unital commutative complex Banach algebra, and  $M(A_e) = \{\varphi_\infty\}$ , where  $\varphi_\infty$  is the continuous multiplicative linear functional on  $A_e$  defined by  $\varphi_\infty(a + \lambda e) = \lambda$  for all  $a + \lambda e \in A_e$ .

(1) Let  $a$  be a non-zero element of  $A$ ,  $V(A_e, a) = \{z \in C : |z| \leq \|a\|\}$  by [2, Remark 3.8], and  $sp(a) = \{\varphi_\infty(a)\} = \{0\}$ , hence  $co(sp(a)) = \{0\} \subsetneq V(A_e, a)$  since  $\|a\| \neq 0$ . Therefore the direct implication of [4, Theorem 2] doesn't hold.

(2) By [1, lemma 1.10.3],  $D(A_e, e)$  is a nonempty weak\* compact convex subset of  $A'_e$ , then  $ext(D(A_e, e)) \neq \emptyset$ . Assume that each extreme normalized state on  $A_e$  is multiplicative, then  $ext(D(A_e, e)) = \{\varphi_\infty\}$ . Let  $a$  be a non-zero element of  $A$ , by [1, Corollary 1.10.15] there exists  $f \in D(A_e, e)$  such that  $f(a) \neq 0 = \varphi_\infty(a)$ . Therefore  $\overline{co}(ext(D(A_e, e))) = \{\varphi_\infty\} \subsetneq D(A_e, e)$ , which contradicts the Krein-Milman Theorem. This shows that [4, theorem 3] is not valid.

**Remark.** Theorems 5 and 6 in [4] are called into question since the authors used [4, Theorem 3] to prove these results.

Let  $(A, \|\cdot\|)$  be a non-unital complex Banach algebra, and let  $A_e = \{a + \lambda e : a \in A, \lambda \in \mathbb{C}\}$  be the unitization of  $A$  with the identity  $e$ . Let  $\|a + \lambda e\|_{op} = \sup\{\|(a + \lambda e)x\|, \|x(a + \lambda e)\| : x \in A, \|x\| \leq 1\}$  for all  $a + \lambda e \in A_e$ ,  $\|\cdot\|_{op}$  is an algebra seminorm on  $A_e$ . We say that  $\|\cdot\|$  is regular if  $\|\cdot\|_{op} = \|\cdot\|$  on  $A$ . If  $\|\cdot\|$  is regular, it is well known that  $(A_e, \|\cdot\|_{op})$  is a complex Banach algebra. In [3], the following question was asked: If  $(A_e, \|\cdot\|_{op})$  is a complex Banach algebra, is the norm  $\|\cdot\|$  regular ?

In [5], A. Orenstein tried to give an answer to this question in the commutative case, but his proof is not correct since it is essentially based on the direct implication of [4, Theorem 2].

## References

- [1] F. F. Bonsall and J. Duncan, Complete normed algebras, New York: Springer Verlag 1973.
- [2] A. K. Gaur and T. Husain, Spatial numerical ranges of elements of Banach algebras, International Journal of Mathematics and Mathematical Sciences, 12(4)(1989), 633-640.
- [3] A. K. Gaur and Z. V. Kovářík, Norms, states and numerical ranges on direct sums, Analysis, 11(2-3)(1991), 155-164.
- [4] F. Golfarshchi and A. A. Khalilzadeh, Numerical radius preserving linear maps on Banach algebras, International Journal of Pure and Applied Mathematics, 88(2)(2013), 233-238.
- [5] A. Orenstein, Regular norm and the operator seminorm on a non-unital complex commutative Banach algebra, arXiv:1410.8790v2 [math.FA] 2015.
- [6] C. E. Rickart, General theory of Banach algebras, New York: Van Nostrand 1960.

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